# Investigation of the generalised Wigner-Dunkl harmonic oscillator and its coherent states

**P. Sedaghatnia**[1,*], **H. Hassanabadi**[1,2,†], **G. Junker**[3,4,‡], **J. Kříž**[2,§], **S. Hassanabadi**[2,¶], **W.S. Chung**[5,‖]

[1] *Faculty of Physics, Shahrood University of Technology, Shahrood, Iran P. O. Box : 3619995161-316.*

[2] *Department of Physics, University of Hradec Králové,Rokitanského 62, 500 03 Hradec Králové, Czechia.*

[3] *European Southern Observatory, Karl-Schwarzschild-Straße 2, 85748 Garching, Germany.*

[4] *Institute for Theoretical Physic I, Friedrich-Alexander University Erlangen-Nuremberg, Staudtstraße 7, 91058 Erlangen, Germany.*

[5] *Department of Physics and Research Institute of Natural Science, College of Natural Science, Gyeongsang National University, Jinju 660-701, Korea.*

**Abstract**

This study investigates the quantum mechanics of the harmonic oscillator in the presence of the Generalised-Wigner-Dunkl formalism based on creation and annihilation operators, which include also the reflection operator. We provide explicit results on the spectral properties of the Hamiltonian and an explicit form of the quantum propagator. Deformed coherent states are introduced and their non-classical properties such as photon statistics, Mandel parameter, and bunching or anti-bunching effect are studied.

**Keywords**: Generalised-Wigner-Dunkl operator; Propagator, Coherent States; Mandel Parameter; (Anti)Bunching Effect.

## 1 Introduction

In recent years, quantum algebras and quantum groups have been the subject of intensive research in several fields of physics and mathematics. In ref. [1] the deformed oscillator was introduced and then further investigated in refs. [2, 3, 4] based on the algebra of creation and destruction operators with q-deformation. In the last few years, there has been a growing interest in the search for deformation, which has found applications in several branches of physics [5, 6, 7, 8, 9, 10, 11, 12, 13]. Of particular interest has been the deformation in quantum mechanics, which is nowadays call Wigner-Dunkl (WD) quantum mechanics. For some articles reviewing the Dunkl oscillator system see, for example, [14, 15, 16, 17]. Wigner-Dunkl quantum mechanics goes back to the original work of Wigner [18], where he proposed a new quantisation method called Wigner algebra. Based on the new quantisation, the product of a constant number (the Wigner or deformation parameter) and the parity operator can be introduced in the standard commutation relation between the momentum and the position operator, which does not change the equation of motion. In ref. [18] Wigner considered the reflection operator in the boson algebra, where he introduced a form of deformation that does not change the Hamiltonian for the harmonic oscillator. Later, in 1989, Charles Dunkl [19] introduced differential-difference operators of the form

$$D_x^{\nu} f(x) = \frac{\partial}{\partial x} f(x) + \nu \left( \frac{f(x) - f(-x)}{x} \right) , \tag{1}$$

where the Dunkl operator $D_x^{\nu}$ includes the reflection operator $R$, with $(Rf)(x) = f(-x)$. Dunkl operators provided a very simple way to construct the differential equations we needed. They have appeared in various

*E-mail: pa.sedaghatnia@gmail.com (Corresponding author)
†E-mail: h.hasanabadi@shahroodut.ac.ir
‡E-mail: gjunker@eso.org, georg.junker@fau.de
§E-mail: jan.kriz@uhk.cz
¶Email: s.hassanabadi@yahoo.com
‖Email: mimip44@naver.com

mathematical physics problems [20, 21, 22, 23]. One of the applications of Dunkl operators is in harmonic analysis, where they become Laplace-Dunkl operators [24]. However, in these investigations the effect of the reflection operator $R$ is not yet fully taken into account. For completeness, we should consider the effect of the reflection operator $R$ also on the first term in (1), which contains the differential operator. For this purpose, we propose a generalised Dunkl partial differential operator of the form

$$\left(D_x^{\alpha,\beta,\gamma} f\right)(x) := \frac{\partial}{\partial x} f(x) + \frac{\alpha}{x} f(x) + \frac{\beta}{x} f(-x) + \gamma \frac{\partial}{\partial x} f(-x)\,, \tag{2}$$

which was also considered in some recent studies [25, 26, 27, 28, 29]. The above generalised Wigner-Dunkl (GWD) derivative (2) was first introduced in [30] and reduces to the standard Wigner-Dunkl derivate in one dimension when $\alpha = -\beta > -\frac{1}{2}$ and $\gamma = 0$.

Recently, in refs. [31, 32] the authors investigated the Schrödinger equation for the standard Dunkl-Coulomb problem in three dimensional space. The two-dimensional problems in the presence of the Dunkl operator for harmonic oscillator and Coulomb potentials have been investigated in refs. [33, 34]. In addition, the authors of ref. [35] have solved the Klein-Gordon equation in the presence of the standard Dunkl operator. More recently the generalised form of the Dunkl operator has attracted some attention. See, for example, refs. [26, 27, 29, 36].

Thermodynamics of various quantum systems in the presence of the Dunkl operator were investigated in the literature, including the study of effects of the Dunkl parameter on these systems. For example thermodynamic properties of Dunkl-bosonic systems in the context of Dunkl-statistical mechanics are discussed in [37, 38, 39, 40]. In ref. [37] the thermodynamics of boson systems related to Dunkl differential–difference operators is presented. The ideal Bose gas and blackbody radiation in the framework of Dunkl formalism have been studied in ref. [38], and ref. [39] deals with the condensation of ideal Bose gas in a gravitational field in the framework of Dunkl-statistics. The thermal properties of relativistic-Dunkl oscillators are investigated in ref. [40], and then the same authors studied the three-dimensional Dunkl-Klein-Gordon equations under Coulomb potential in [41]. A study on Dunkl graphene in a constant magnetic field can be found in ref. [42].

The main goal of this paper is to show that the generalised Wigner-Dunkl Schrödinger equation for a harmonic oscillator potential in one dimension is exactly solvable. Also, we are going to introduce the deformed-number operator, which is closely related to the GWD harmonic oscillator Hamiltonian. Then, we show how the coherent states for the deformed boson algebra are obtained.

This paper is organised as follows. In section 2, we describe the Hamiltonian of a harmonic oscillator in the presence of the GWD formalism, by introducing deformed creation and annihilation operators. Then we derive the deformed Heisenberg relation depending on the parity operator and the GWD parameters. In section 3, we explicitly solve the spectral problem of the GWD harmonic oscillator and introduce the number operator representing the number of photons. We also present an explicit expression for the quantum propagator and discuss the minimum uncertainty in GWD quantum mechanics. Then, in section 4, we construct the deformed coherent states as eigenstates of the deformed annihilation operator. Section 5 discusses some non-classical properties of these coherent states. Finally, in section 6 we summarise our findings.

## 2 Generalised Wigner-Dunkl Formalism

The Hamiltonian of the one-dimensional harmonic oscillator in GWD quantum mechanics is, in units where Planck's constant $\hbar$, the oscillator frequency $\omega$ and the mass $m$ of the particle are given $\hbar = \omega = m = 1$, defined by [43, 44]

$$H := -\frac{1}{2} D_x^2 + \frac{x^2}{2}. \tag{3}$$

Here $x \in \mathbb{R}$ and $D_x$ denote the position and GWD derivative, respectively. The latter is explicitly given by [25, 26, 27, 29]

$$D_x := \frac{\partial}{\partial x} + \frac{\alpha}{x} + \frac{\beta}{x} R - \gamma R \frac{\partial}{\partial x}\,, \tag{4}$$

where $R$ is the parity operator, that is, it obeys the following condition.

$$(Rf)(x) = f(-x)\,, \qquad R^2 = 1. \tag{5}$$

Here we have omitted the superscripts $\alpha, \beta, \gamma$ on the GWD derivative in order to keep the notation simple. These operators act on a weighted Hilbert space $\mathcal{H} := L^2(\mathbb{R}, \mathrm{d}\mu)$ with measure

$$\mathrm{d}\mu(x) := \mathrm{d}x\, |x|^{\frac{2(\alpha-\beta\gamma)}{1-\gamma^2}}\,. \tag{6}$$

For a well-defined weight function, the real deformation parameters $\alpha$, $\beta$ and $\gamma$ shall be restricted such that

$$\mu := \frac{\alpha - \beta\gamma}{1-\gamma^2} > -\frac{1}{2}\,. \tag{7}$$

Obviously, $|\gamma| = 1$ needs to be excluded, too. This leads us to the two disjunct regions $|\gamma| < 1$ and $|\gamma| > 1$. Only the first one contains the standard Wigner-Dunkl case $\gamma = 0$. In addition, as we will see below, the second region will not result in a proper ground state (22). As a result we have the additional condition

$$|\gamma| < 1\,. \tag{8}$$

With the above measure the scalar product of two states $\phi, \psi \in \mathcal{H}$ is then defined as

$$(\phi, \psi) := \int_{\mathbb{R}} \mathrm{d}x\, |x|^{2\mu}\, \phi^*(x)\psi(x)\,. \tag{9}$$

Let us consider two arbitrary states $\phi, \psi \in \mathcal{H}$ and the matrix element

$$\langle\phi|\partial_x\psi\rangle = -\int_{-\infty}^{+\infty} \mathrm{d}x\, |x|^{2\mu}\psi(x) \left( (\partial_x\phi^*)(x) + \frac{2\mu}{x}\phi^*(x) \right)\,, \tag{10}$$

which is explicitly calculated in the appendix. Then we can conclude that the adjoint of the partial derivative with respect to $x$ in the weighted Hilbert space $\mathcal{H}$ is given by $(\partial_x)^\dagger = -\partial_x - \frac{2\mu}{x}$. Using this result together with $R^\dagger = R$ and $x^\dagger = x$ we find for the adjoint $D_x^\dagger$ following form

$$D_x^\dagger = -\left(\frac{1+\gamma R}{1-\gamma R}\right) D_x\,, \qquad (1+\gamma R)D_x = -D_x^\dagger(1+\gamma R). \tag{11}$$

Again more details are given in the appendix.

This allows us to introduce ladder operators $a$ and $a^\dagger$ given by [43]

$$a := \frac{1}{\sqrt{2}}\left(x + \frac{1+\gamma R}{\sqrt{1-\gamma^2}} D_x\right)\,, \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{1+\gamma R}{\sqrt{1-\gamma^2}} D_x\right). \tag{12}$$

Then, the commutation relation between $a$ and $a^\dagger$ in the GWD formalism is [25, 26, 30]

$$\left[a, a^\dagger\right] = \sqrt{1-\gamma^2}\,(1 + 2\nu R)\,, \tag{13}$$

which follows from the relations

$$[D_x, x] = 1 - 2\beta R - \gamma R\{\partial_x, x\}\,, \qquad \{D_x, x\} = 2\alpha - \gamma R + \{\partial_x, x\}\,, \tag{14}$$

where the curl brackets denote the usual anti-commutator. In (13), in addition, we have introduced the parameter

$$\nu := \frac{\alpha\gamma - \beta}{1-\gamma^2} = \gamma\mu - \beta, \tag{15}$$

for convenience.

Furthermore, a little calculation leads us to the commutation relation with the Hamiltonian (3) given by

$$[H,a] = -\left\{\sqrt{\frac{1-\gamma^2}{2}}x + \frac{1+\gamma R}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\alpha}{x} + \frac{\beta}{x}R - \gamma R\frac{\partial}{\partial x}\right)\right\} = -\sqrt{1-\gamma^2}\,a, \tag{16}$$

and

$$[H,a^\dagger] = \left\{\sqrt{\frac{1-\gamma^2}{2}}x - \frac{1+\gamma R}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{\alpha}{x} + \frac{\beta}{x}R - \gamma R\frac{\partial}{\partial x}\right)\right\} = \sqrt{1-\gamma^2}\,a^\dagger\,. \tag{17}$$

That is, the two operators (12) indeed act as raising and lowering operators on eigenstates of $H$, which can be put into the form

$$H = \frac{1}{2}\left(a^\dagger a + aa^\dagger\right)\,. \tag{18}$$

In concluding this section we also note that

$$\{a,R\} = 0\,, \qquad [H,R] = 0\,, \tag{19}$$

and therefore, $H$ and $R$ can exhibit joint eigenstates to be investigated in the following section.

## 3 Spectral properties of the GWD harmonic oscillator

To begin with let us consider two functions $f^\pm$ being even and odd, respectively. That is, $Rf^\pm = \pm f^\pm$. Then we find the obvious relations

$$D_x f^\pm(x) = (1\pm\gamma)\left(\partial_x + \frac{\mu\mp\nu}{x}\right)f^\pm(x)\,. \tag{20}$$

This allows us to interpret the three parameters $\gamma$, $\mu$ and $\nu$ as follows. First, $\gamma$ can be seen as a pure scaling parameter for the $x$-coordinate, the energy and time. This will become more transparent in our way forward. That is, length always appears in the combination $x(1-\gamma^2)^{-1/4}$, energy always scales as $E(1-\gamma^2)^{-1/2}$ and time as $\tau(1-\gamma^2)^{1/2}$, when we choose $\{\mu,\nu,\gamma\}$ as the set of independent deformation parameters, which we will do form now on. Second, the parameter $\mu$ might be viewed as a "gauge" parameter, as via the replacement $\psi(x)\to\tilde{\psi}(x) := |x|^{-\mu}\psi(x)$ we have $\partial_x\tilde{\psi}(x) = |x|^{-\mu}(\partial_x - \frac{\mu}{x})\psi(x)$. Here $\tilde{\psi}\in L^2(\mathbb{R})$ lives in the usual Hilbert space. That is, without loss of generality one could set $\mu = 0$ and consider the Wigner-Dunkl oscillator with remaining deformation parameter $\nu > -1/2$ as discussed, for example, in ref. [45].

The ladder operators (12) act as raising and lowering operators on eigenstates of the Hamiltonian (18), which is obviously bounded from below. In other words, the ground state $\phi_0^\pm$ must be annihilated by $a$, which a priory may be an even and/or odd state,

$$a\phi_0^\pm = 0 \qquad \text{with} \qquad R\phi_0^\pm = \pm\phi_0^\pm\,. \tag{21}$$

This leads us to the two solutions

$$\phi_0^+ = N_+\frac{1}{|x|^{\mu-\nu}}\exp\left\{-\frac{x^2}{2\sqrt{1-\gamma^2}}\right\}\,, \qquad \phi_0^- = N_-\frac{x}{|x|^{\mu+\nu+1}}\exp\left\{-\frac{x^2}{2\sqrt{1-\gamma^2}}\right\}, \tag{22}$$

with normalisation constants $N_\pm$ given by

$$N_\pm^{-2} := \left(1-\gamma^2\right)^{\frac{1}{4}\pm\frac{\nu}{2}}\Gamma\left(\tfrac{1}{2}\pm\nu\right)\,. \tag{23}$$

As concluded already above, $\gamma$ is a scaling factor and $\mu$ acts as a gauge parameter. Hence, with the previous conclusion $\nu > -1/2$ we have to discard the odd solution $\phi_0^-$ and remain with the even ground state $\phi_0 \equiv \phi_0^+$.

The remaining eigenstates may be obtained from the even ground state by applying the raising operator. We note here that application of operator $a$ as well as $a^\dagger$ change the parity of the wave functions each time. That is, the eigenstates obey the relations

$$H\phi_n = E_n\phi_n\,, \qquad R\phi_n = (-1)^n\phi_n\,, \qquad n = 0,1,2,3,\ldots\,. \tag{24}$$

The eigenvalues are explicitly given by

$$E_n = \sqrt{1-\gamma^2}\left(n+\nu+\frac{1}{2}\right), \tag{25}$$

and the corresponding eigenstates are found to be given by

$$\phi_{2m}(x) = N_{2m}\,|x|^{\nu-\mu} L_m^{\nu-\frac{1}{2}}\left(\frac{x^2}{\sqrt{1-\gamma^2}}\right)\exp\left\{-\frac{x^2}{2\sqrt{1-\gamma^2}}\right\}, \tag{26}$$

$$\phi_{2m+1}(x) = N_{2m+1}\,|x|^{\nu-\mu} x L_m^{\nu+\frac{1}{2}}\left(\frac{x^2}{\sqrt{1-\gamma^2}}\right)\exp\left\{-\frac{x^2}{2\sqrt{1-\gamma^2}}\right\}. \tag{27}$$

Here $L_p^\varepsilon(z) := \frac{\Gamma(p+\varepsilon+1)}{\Gamma(\varepsilon+1)n!}\,{}_1F_1(-p,\varepsilon+1,z)$ stands for the associated Laguerre polynomial of order $p$. The normalisation constants are given by

$$N_{2m}^{-2} := \left(1-\gamma^2\right)^{\frac{\nu}{2}+\frac{1}{4}}\frac{\Gamma(m+\nu+\frac{1}{2})}{m!}, \qquad N_{2m+1}^{-2} := \left(1-\gamma^2\right)^{\frac{\nu}{2}+\frac{3}{4}}\frac{\Gamma(m+\nu+\frac{3}{2})}{m!}. \tag{28}$$

The above results are obtain via relations (20) resulting in the explicit actions of the ladder operators on these eigenstates given by

$$a\,\phi_{2m} = -\left(1-\gamma^2\right)^{\frac{1}{4}}\sqrt{2m}\,\phi_{2m-1}, \qquad a\,\phi_{2m+1} = \left(1-\gamma^2\right)^{\frac{1}{4}}\sqrt{2m+2\nu+1}\,\phi_{2m}, \tag{29}$$

$$a^\dagger\,\phi_{2m} = \left(1-\gamma^2\right)^{\frac{1}{4}}\sqrt{2m+2\nu+1}\,\phi_{2m+1}, \qquad a^\dagger\,\phi_{2m+1} = -\left(1-\gamma^2\right)^{\frac{1}{4}}\sqrt{2m+2}\,\phi_{2m+2}. \tag{30}$$

These expressions can be unified when using the Dunkl number

$$[n]_\nu := n + \nu\left(1-(-1)^n\right), \tag{31}$$

into the form

$$a\,\phi_n = (-1)^{n-1}\left(1-\gamma^2\right)^{\frac{1}{4}}\sqrt{[n]_\nu}\,\phi_{n-1}, \qquad a^\dagger\,\phi_n = (-1)^n\left(1-\gamma^2\right)^{\frac{1}{4}}\sqrt{[n+1]_\nu}\,\phi_{n+1}. \tag{32}$$

From this we also obtain the relation

$$\phi_n = (-1)^{\frac{n}{2}(n-1)}\frac{(1-\gamma^2)^{-\frac{n}{4}}}{\sqrt{[n]_\nu!}}\left(a^\dagger\right)^n\phi_0, \tag{33}$$

where

$$[n]_\nu\,! := [1]_\nu[2]_\nu\cdots[n]_\nu, \qquad [0]_\nu! := 1. \tag{34}$$

In order to simplify expressions while going forward, we will use Dirac's ket notation and consider eigenstates with a shifted phase as follows

$$|n\rangle := (-1)^{\frac{n}{2}(n-1)}\phi_n. \tag{35}$$

They obviously obey similar relations as the $\phi_n$ do, but without the additional phase factors

$$a|n\rangle = \left(1-\gamma^2\right)^{\frac{1}{4}}\sqrt{[n]_\nu}\,|n-1\rangle, \quad a^\dagger|n\rangle = \left(1-\gamma^2\right)^{\frac{1}{4}}\sqrt{[n+1]_\nu}\,|n+1\rangle, \tag{36}$$

$$|n\rangle = \frac{(1-\gamma^2)^{-\frac{n}{4}}}{\sqrt{[n]_\nu!}}\left(a^\dagger\right)^n|0\rangle, \tag{37}$$

and are simultaneous eigenstates of the Hamiltonian (18) and the number operator $N$, which may be defined via the Hamiltonian as

$$N := \frac{H}{\sqrt{1-\gamma^2}} - \left(\nu+\frac{1}{2}\right), \qquad N|n\rangle = n|n\rangle, \qquad n\in\mathbb{N}_0. \tag{38}$$

The above results do not come as a surprise. As pointed out in the beginning of this section, the generalised Wigner-Dunkl oscillator is closely related to the standard Wigner-Dunkl oscillator as discussed by various authors [17, 46]. Indeed, the spectral properties (25-27) found here are in form identical with those of the standard oscillator. That is, the even and odd eigenfunctions represent in essence the generalised Hermite polynomials as defined by Rosenblum [47], however with a rescaled coordinate given by $z = x/(1-\gamma^2)^{1/4}$. In addition, the deformation parameter $\nu$ is given here by eq. (15), whereas in the standard case it is $\alpha = -\beta$ with $\gamma = 0$. Let us remark that we switched from parameters $\alpha$ and $\beta$ to the new set of independent parameters $\mu$ and $\nu$ with

$$\alpha = \mu - \gamma\nu\,, \qquad \beta = \mu\gamma - \nu\,, \qquad \mu,\nu > -\frac{1}{2}\,, \qquad |\gamma| < 1\,, \tag{39}$$

which are more appropriate as $\mu$ in essence characterises the weighted Hilbert space, whereas $\nu$ acts as the deformation parameter and $\gamma$ as a simple scaling factor. In that sense, the current results are a generalisation of the standard Wigner-Dunkl oscillator results.

Above spectral properties allow us now to take a closer look at the dynamics of the GWD harmonic oscillator. In doing so we consider the propagator generated by Hamiltonian (3)

$$K_\nu(x,y;\tau) := \langle x|\mathrm{e}^{-\mathrm{i}H\tau}|y\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{-\mathrm{i}E_n\tau}\phi_n(x)\phi_n^*(y)\,. \tag{40}$$

Let us consider the even and odd parts in above sum separately. First we evaluate the even sum by setting $\varphi := \tau\sqrt{1-\gamma^2}$, $\xi := x(1-\gamma^2)^{-\frac{1}{4}}$ and $\eta := y(1-\gamma^2)^{-\frac{1}{4}}$,

$$\sum_{m=0}^{\infty} \mathrm{e}^{-\mathrm{i}E_{2m}\tau}\phi_{2m}(x)\phi_{2m}^*(y) = \frac{|\xi\eta|^\nu \mathrm{e}^{-\mathrm{i}\varphi(\nu+\frac{1}{2})}}{|xy|^\mu\,(1-\gamma^2)^{\frac{1}{4}}}\exp\left\{-\frac{\xi^2+\eta^2}{2}\right\}\sum_{m=0}^{\infty}\frac{m!\,\mathrm{e}^{-2\mathrm{i}m\varphi}}{\Gamma(m+\nu+\frac{1}{2})}L_m^{\nu-\frac{1}{2}}(\xi^2)L_m^{\nu-\frac{1}{2}}(\eta^2)\,. \tag{41}$$

The above sum is known as Hille-Hardy formula in the literature [48] and results in below expression including the modified Bessel function of first kind.

$$\sum_{m=0}^{\infty} \mathrm{e}^{-\mathrm{i}E_{2m}\tau}\phi_{2m}(x)\phi_{2m}^*(y) = \frac{|xy|^{\frac{1}{2}-\mu}}{\sqrt{1-\gamma^2}}\frac{1}{2\mathrm{i}\sin\varphi}\exp\left\{\frac{\mathrm{i}}{2}(\xi^2+\eta^2)\cot\varphi\right\}\mathrm{I}_{\nu-\frac{1}{2}}\left(\frac{|\xi\eta|}{\mathrm{i}\sin\varphi}\right)\,. \tag{42}$$

For the odd sum we proceed similarly and arrive at

$$\sum_{m=0}^{\infty} \mathrm{e}^{-\mathrm{i}E_{2m+1}\tau}\phi_{2m+1}(x)\phi_{2m+1}^*(y) = \frac{|xy|^{\frac{1}{2}-\mu}}{\sqrt{1-\gamma^2}}\frac{\mathrm{sgn}\,(xy)}{2\mathrm{i}\sin\varphi}\exp\left\{\frac{\mathrm{i}}{2}(\xi^2+\eta^2)\cot\varphi\right\}\mathrm{I}_{\nu+\frac{1}{2}}\left(\frac{|\xi\eta|}{\mathrm{i}\sin\varphi}\right)\,. \tag{43}$$

Adding both provides us with a closed-form expression for the propagator of the GWD harmonic oscillator

$$K_\nu(x,y;\tau) = |xy|^{-\mu}\frac{\sqrt{|\xi\eta|}}{2\mathrm{i}\sin\varphi}\exp\left\{\frac{\mathrm{i}}{2}(\xi^2+\eta^2)\cot\varphi\right\}\left[\mathrm{I}_{\nu-\frac{1}{2}}\left(\frac{|\xi\eta|}{\mathrm{i}\sin\varphi}\right)+\mathrm{sgn}\,(\xi\eta)\,\mathrm{I}_{\nu+\frac{1}{2}}\left(\frac{|\xi\eta|}{\mathrm{i}\sin\varphi}\right)\right]\,, \tag{44}$$

which can, with the help of the deformed exponential function defined below in (55), be put into the form

$$K_\nu(x,y;\tau) = \frac{|xy|^{-\mu}}{\Gamma(\nu+\frac{1}{2})}\frac{1}{\sqrt{2\mathrm{i}\sin\varphi}}\left(\frac{|\xi\eta|}{2\mathrm{i}\sin\varphi}\right)^\nu\exp\left\{\frac{\mathrm{i}}{2}(\xi^2+\eta^2)\cot\varphi\right\}\mathrm{E}_\nu\left(\frac{\xi\eta}{\mathrm{i}\sin\varphi}\right)\,. \tag{45}$$

In the special case $\nu = 0$ this reduces to the well-known expression for the harmonic oscillator propagator in the weighted Hilbert space $\mathcal{H}$

$$K_0(x,y;\tau) = \frac{|xy|^{-\mu}}{\sqrt{2\pi\mathrm{i}\sin\varphi}}\exp\left\{\frac{\mathrm{i}}{2}(\xi^2+\eta^2)\cot\varphi - \mathrm{i}\frac{\xi\eta}{\sin\varphi}\right\}\,. \tag{46}$$

As a final comment let us consider the variance of the position operator $x = \frac{1}{\sqrt{2}}(a+a^\dagger)$ and the self-adjoint momentum operator, cf. (11),

$$p := -\mathrm{i}\frac{1-\gamma R}{\sqrt{1-\gamma^2}}\,D_x = -\frac{\mathrm{i}}{\sqrt{2}}(a-a^\dagger), \tag{47}$$

which results in

$$\Delta x_n^2 := \langle n|x^2|n\rangle - (\langle n|x|n\rangle)^2 = E_n\,, \qquad \Delta p_n^2 := \langle n|p^2|n\rangle - (\langle n|p|n\rangle)^2 = E_n\,. \tag{48}$$

This leads us to the uncertainty relation

$$\Delta x_n \Delta p_n = E_n = \sqrt{1-\gamma^2}\left(n+\nu+\frac{1}{2}\right) \geq \sqrt{1-\gamma^2}\left(\tfrac{1}{2}+\nu\right)\,. \tag{49}$$

Therefore, we may set $\gamma = 0$ without loss of generality as it is a scaling factor, and conclude that for $\nu > 0$ the minimum uncertainty in GWD quantum mechanics is larger than that in ordinary quantum mechanics, whereas for $-\frac{1}{2} < \nu < 0$, the deformed minimum uncertainty is smaller. For $\nu = 0$ we get the usual undeformed uncertainty relation of the harmonic oscillator.

## 4 Coherent states in the GWD formalism

In this section we are studying the coherent states within the generalised Wigner-Dunkl formalism. In doing so we define the coherent states as eigenstates of the annihilation operator, [43, 44, 49, 50]

$$a|\lambda\rangle = \lambda|\lambda\rangle. \tag{50}$$

Here, $\lambda \in \mathbb{C}$ is an arbitrary complex number. Expanding those states into the complete set of eigenstates (35)

$$|\lambda\rangle = \sum_{n=0}^{\infty} C_n|n\rangle\,, \tag{51}$$

we obtain the recursion relation

$$C_{n+1} = \frac{\lambda}{(1-\gamma^2)^{1/4}\sqrt{[n+1]_\nu}}C_n. \tag{52}$$

This brings us to the decomposition

$$|\lambda\rangle = C_0(\lambda)\sum_{n=0}^{\infty}\frac{\lambda^n}{(1-\gamma^2)^{n/4}\sqrt{[n]_\nu!}}|n\rangle\,. \tag{53}$$

Here $C_0$ is a normalisation constant given by

$$C_0^{-2}(\lambda) := \mathrm{E}_\nu\left(\frac{|\lambda|^2}{\sqrt{1-\gamma^2}}\right)\,, \tag{54}$$

where $\mathrm{E}_\nu$ is a deformed exponential function defined by

$$\mathrm{E}_\nu(z) := \sum_{n=0}^{\infty}\frac{z^n}{[n]_\nu!}\,. \tag{55}$$

More precisely, $\mathrm{E}_\nu$ is the Dunkl kernel associated with the reflection group $Z_2$ generate by $R$. In order to see that, we first state the relations

$$[2m]_\nu! = 2^{2m}m!\left(\nu+\tfrac{1}{2}\right)_m\,, \qquad [2m+1]_\nu! = 2^{2m+1}m!(\nu+\tfrac{1}{2})\left(\nu+\tfrac{3}{2}\right)_m\,, \tag{56}$$

and observe that

$$\mathrm{E}_\nu(z) = \sum_{m=0}^{\infty}\frac{(z/2)^{2m}}{m!\left(\nu+\frac{1}{2}\right)_m} + \frac{z}{2\nu+1}\sum_{m=0}^{\infty}\frac{(z/2)^{2m}}{m!\left(\nu+\frac{3}{2}\right)_m} = {}_0F_1(\nu+\tfrac{1}{2},\tfrac{z^2}{4}) + \frac{z}{2\nu+1}\,{}_0F_1(\nu+\tfrac{3}{2},\tfrac{z^2}{4})\,. \tag{57}$$

These expressions are in essence identical to the modified Bessel functions $\mathrm{I}_{\nu\pm\frac{1}{2}}$ of first kind with index $\nu \pm \frac{1}{2}$, recall that $\mathrm{I}_\varepsilon(z) = \frac{(z/2)^\varepsilon}{\Gamma(\varepsilon+1)}\, {}_0F_1(\varepsilon+1, \frac{z^2}{4})$, and allow us to rewrite the deformed exponential function as follows

$$\mathrm{E}_\nu(z) = \Gamma(\nu + \tfrac{1}{2}) \left(\frac{2}{z}\right)^{\nu-\frac{1}{2}} \left[\mathrm{I}_{\nu-\frac{1}{2}}(z) + \mathrm{I}_{\nu+\frac{1}{2}}(z)\right] . \tag{58}$$

Obviously for $\nu = 0$ this becomes the usual exponential function $\mathrm{E}_0(z) = \mathrm{e}^z$. As a side remark let us mention that the deformed exponential function may also be expressed in terms of so-called normalised spherical Bessel functions $j_\varepsilon(u) := {}_0F_1(\varepsilon+1, -\frac{u^2}{4})$ as defined by Rösler [51],

$$\mathrm{E}_\nu(z) = j_{\nu-\frac{1}{2}}(\mathrm{i}z) + \frac{z}{2\nu+1} j_{\nu+\frac{1}{2}}(\mathrm{i}z) . \tag{59}$$

The coherent states are independent of the deformation parameter $\mu$ and therefore they are identical in form with those obtained by Ghazouani [52] for the standard Wigner-Dunkl oscillator when setting $\gamma = 0$. Nevertheless, our results are different in various aspects. Here the annihilation and creation operators obey a more general commutation relation (13) involving the $\gamma$-dependent scaling factor not being present in standard Wigner-Dunkl oscillator. In addition, the parameter $\nu$ characterising the deformation is in general different to the parameter $\mu$ characterising the underlying weighted Hilbert space. Both parameters are identical when considering the standard Wigner-Dunkl oscillator. Note that in [52] the authors only considers positive values of the deformation parameter $\nu$. However, here we also allow negative values as we restrict the deformation parameter by $\nu > -\frac{1}{2}$. In fact, the range $-\frac{1}{2} < \nu < 0$ has some additional features to offer as we have already seen in the previous section when discussing the uncertainly relation and as we will see below when discussing the expectation value of $R$. Nevertheless, one can extend several results from the standard case as presented in [52] to the current investigation bearing in mind the difference in the deformation parameters.

According to the general principles as laid out by Klauder and Skagerstam [53], coherent states should obey a set of minimal requirements, which are (i) strong continuity in the label $\lambda$ and (ii) the resolution of unity in $\mathcal{H}$.

The proof of strong continuity goes along the same line as for the standard WD harmonic oscillator [52]. Noting that the overlap of two coherent states, say $|\kappa\rangle$ and $|\lambda\rangle$, is given by

$$\langle\kappa|\lambda\rangle = C_0(\kappa)C_0(\lambda)\mathrm{E}_\nu\left(\frac{\kappa^*\lambda}{\sqrt{1-\gamma^2}}\right) = \frac{\mathrm{E}_\nu\left(\frac{\kappa^*\lambda}{\sqrt{1-\gamma^2}}\right)}{\sqrt{\mathrm{E}_\nu\left(\frac{|\kappa|^2}{\sqrt{1-\gamma^2}}\right)\mathrm{E}_\nu\left(\frac{|\lambda|^2}{\sqrt{1-\gamma^2}}\right)}} , \tag{60}$$

the strong continuity immediately follows from that of the deformed exponential function. We note here that the overlap (60) is often called the reproducing kernel and demonstrates the well-known over-completeness of coherent states.

At this point we shall also note that $R|\lambda\rangle = |-\lambda\rangle$, which immediately follows from relation (53). Hence, the expectation value of the reflection operator in a coherent state reads

$$\langle\lambda|R|\lambda\rangle = \langle\lambda|-\lambda\rangle = \frac{\mathrm{E}_\nu(-\rho)}{\mathrm{E}_\nu(\rho)} = \frac{\mathrm{I}_{\nu-\frac{1}{2}}(\rho) - \mathrm{I}_{\nu+\frac{1}{2}}(\rho)}{\mathrm{I}_{\nu-\frac{1}{2}}(\rho) + \mathrm{I}_{\nu+\frac{1}{2}}(\rho)} , \tag{61}$$

where we have set $\rho := \frac{|\lambda|^2}{\sqrt{1-\gamma^2}}$. For $\nu \geq 0$ it is known ** that $0 < \mathrm{I}_{\nu+\frac{1}{2}}(\rho)/\mathrm{I}_{\nu-\frac{1}{2}}(\rho) < 1$ for $\rho > 0$ and conclude that $0 < \langle\lambda|R|\lambda\rangle < 1$ for positive deformation parameter $\nu$. The expectation value (61) depends only on $|\lambda|$ and behaves for small and large $\lambda$ as follows

$$\langle\lambda|R|\lambda\rangle = 1 - \tfrac{\rho}{\nu+\frac{1}{2}} + O(\rho^2) , \qquad \langle\lambda|R|\lambda\rangle = \tfrac{\nu}{2\rho} + O(\rho^{-2}) . \tag{62}$$

**This follows from Theorem 1 in [54], which has an even sharper bound.

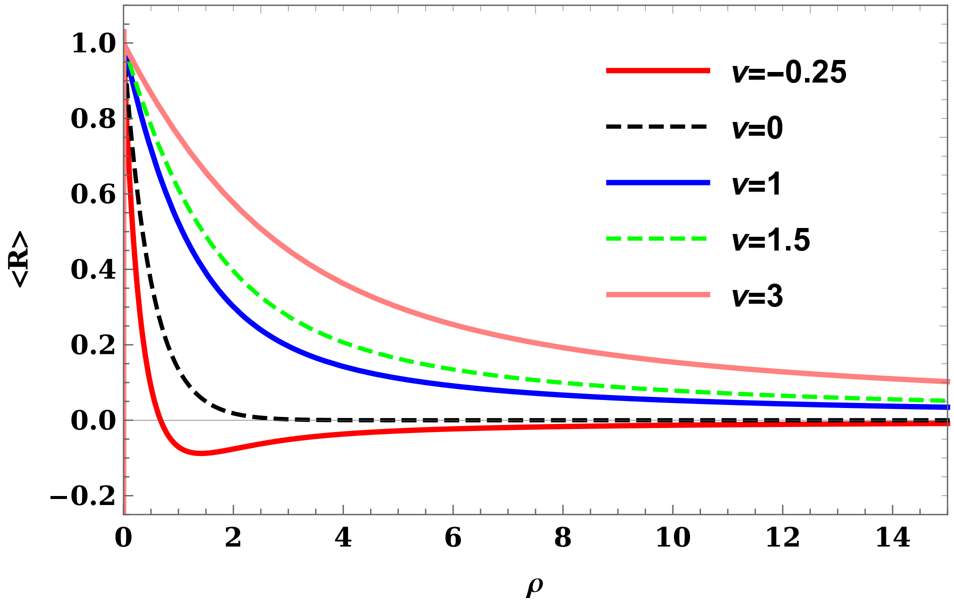


Figure 1: The expectation value of the reflection operator $\langle R\rangle \equiv \langle\lambda|R|\lambda\rangle$ as a function of $\rho = |\lambda|^2/\sqrt{1-\gamma^2}$ for various values of the deformation parameter $\nu$.

Figure 1 shown graphs of the expectation value (61) for various values of the deformation parameter $\nu$ as a function of $\rho$, confirming our above conclusions. First, we note that for non-negative deformation parameter $\nu \geq 0$ we indeed see $0 < \langle\lambda|R|\lambda\rangle < 1$ when $\rho > 0$. We also conclude that for those deformation parameters this expectation value is a strictly decreasing function in $\rho$. In the case of a negative deformation $-\frac{1}{2} < \nu < 0$ the behaviour is differed as now the expectation value becomes negative at a given finite $\rho$, assumes a minium value, and finally approaches the zero value from below for $\rho \to \infty$, in line with the asymptotic behavior (62). We also observe that the various graphs do not intersect, that is, for a fixed value of $\rho$ the expectation value (61) is a stricly increasing function of the deformation parameter $\nu$.

As for the resolution of unity let us refer to the work by Ghazouani [52], where the corresponding measures for the even and odd subspaces are constructed explicitly. These results, although derived for the standard WD case, may be carried over to the current case of the GWD study. This is also the case for several more properties of the GWD coherent states. Hence, we limit ourselves to highlight the differences.

First, we consider the expectation value of the Hamiltonian in a coherent state, which is given by

$$\begin{aligned}\langle\lambda|H|\lambda\rangle &= \tfrac{1}{2}\langle\lambda|a^\dagger a + aa^\dagger|\lambda\rangle \\ &= \langle\lambda|a^\dagger a|\lambda\rangle + \tfrac{1}{2}\langle\lambda|[a,a^\dagger]|\lambda\rangle \\ &= |\lambda|^2 + \tfrac{1}{2}\sqrt{1-\gamma^2}\,(1+2\nu\langle\lambda|R|\lambda\rangle)\,.\end{aligned} \tag{63}$$

Second, we consider position and momentum operator and arrive at relations

$$\begin{aligned}\langle\lambda|x|\lambda\rangle &= \tfrac{1}{\sqrt{2}}(\lambda+\lambda^*), \\ \langle\lambda|x^2|\lambda\rangle &= \tfrac{1}{2}\langle\lambda|(a+a^\dagger)^2|\lambda\rangle = \langle\lambda|H|\lambda\rangle + \tfrac{1}{2}\left(\lambda^2+\lambda^{*2}\right), \\ \langle\lambda|p|\lambda\rangle &= \tfrac{-i}{\sqrt{2}}(\lambda-\lambda^*), \\ \langle\lambda|p^2|\lambda\rangle &= \langle\lambda|H|\lambda\rangle - \tfrac{1}{2}\left(\lambda^2+\lambda^{*2}\right).\end{aligned} \tag{64}$$

This implies for the standard deviations within a coherent state the relations

$$(\Delta p_\lambda)^2 := \langle\lambda|p^2|\lambda\rangle - (\langle\lambda|p|\lambda\rangle)^2 = \langle\lambda|H|\lambda\rangle - |\lambda|^2\,, \tag{65}$$

$$(\Delta x_\lambda)^2 := \langle\lambda|x^2|\lambda\rangle - (\langle\lambda|x|\lambda\rangle)^2 = \langle\lambda|H|\lambda\rangle - |\lambda|^2\,, \tag{66}$$

leading us to the uncertainty relation for the coherent states

$$\Delta x_\lambda \Delta p_\lambda = \langle\lambda|H|\lambda\rangle - |\lambda|^2 = \sqrt{1-\gamma^2}\left(\tfrac{1}{2}+\nu\langle\lambda|R|\lambda\rangle\right) . \tag{67}$$

This result coincides in the limit $\gamma = 0$ with the result [52] obtained for the standard WD coherent states.

Noting that $[p,x] = \mathrm{i}[a^\dagger, a]$, the Robertson uncertainty relation for an arbitrary state $|\psi\rangle \in \mathcal{H}$ implies

$$\Delta x_\psi \Delta p_\psi \geq \tfrac{1}{2}\left|\langle\psi|[a,a^\dagger]|\psi\rangle\right| = \sqrt{1-\gamma^2}\left|\tfrac{1}{2}+\nu\langle\psi|R|\psi\rangle\right| , \tag{68}$$

which demonstrates that the coherent states are minimal uncertainty states. Here let us note that the relation (67) generalises the relation found for the standard WD coherent states as, in addition, it contains the $\gamma$-dependent prefactor. Only for $\gamma = 0$ eq. (67) reduces to the one obtained in [52].

Finally let us also consider the time evolution of the coherent state,

$$|\lambda(\tau)\rangle := \mathrm{e}^{-\mathrm{i}\tau H}|\lambda\rangle = C_0(\lambda)\sum_{n=0}^{\infty}\frac{\lambda^n \mathrm{e}^{-\mathrm{i}\tau E_n}}{(1-\gamma^2)^{n/4}\sqrt{[n]_\nu!}}|n\rangle = \mathrm{e}^{-\mathrm{i}\tau\sqrt{1-\gamma^2}(\nu+\frac{1}{2})}|\mathrm{e}^{-\mathrm{i}\tau\sqrt{1-\gamma^2}}\lambda\rangle , \tag{69}$$

which is the expected result. That is, up to a trivial phase, a coherent states remains to be a coherent state and oscillates under time evolution, $\lambda(t) = \mathrm{e}^{-\mathrm{i}\tau\sqrt{1-\gamma^2}}\lambda(0)$, with a periodicity given by $T = \frac{2\pi}{\sqrt{1-\gamma^2}}$.

# 5 Non-classical Properties of GWD-deformed Coherent States

In this section, we discuss some non-classical properties of GWD-deformed coherent states [43, 49, 50]. To begin with, let us consider the photon statistics. That is the probability distribution to find $n$ photons within a coherent state defined by

$$P_n^{(\nu)} := |\langle n|\lambda\rangle|^2 = \mathrm{E}_\nu^{-1}\left(\frac{|\lambda|^2}{\sqrt{1-\gamma^2}}\right)\frac{|\lambda|^{2n}}{[n]_\nu!} , \tag{70}$$

which is not a Poissonian distribution due to the presence of the deformation parameter $\nu$. Only in the case $\nu = 0$ the Poisson distribution is obtained. For $-\frac{1}{2} < \nu < 0$ we obviously have $[n]_\nu \leq n$ and therefore $P_n^{(\nu)} \geq P_n^{(0)}$. On the other hand, for $\nu > 0$ follows $[n]_\nu \geq n$ and $P_n^{(\nu)} \leq P_n^{(0)}$.

In the following we will consider various non-classical properties exhibited by the GWD coherent states. In order to keep expressions simple we will use the notation $\langle\cdot\rangle := \langle\lambda|\cdot|\lambda\rangle$ for expectation values in a coherent state. The first one we look at is the expectation value of the number operator (38), which can be expressed in terms of the expectation value of the Hamiltonian

$$\langle N\rangle = \langle\lambda|N|\lambda\rangle = \sum_{n=0}^{\infty} nP_n^{(\nu)} = \frac{|\lambda|^2}{\sqrt{1-\gamma^2}} + \nu\left(\langle\lambda|R|\lambda\rangle - 1\right) = \rho - \nu + \nu\langle R\rangle . \tag{71}$$

As in the previous section, we are using here the variable $\rho = \frac{|\lambda^2|}{\sqrt{1-\gamma^2}}$. This expectation value behaves for small and large $\rho$ as follows

$$\langle N\rangle = \frac{\rho}{2\nu+1} + O(\rho^2) , \qquad \langle N\rangle = \rho - \nu + O(\rho^{-1}) , \tag{72}$$

clearly showing the different behavior for negative and positive deformation parameter $\nu$, respectively. Figure 2 shows the expectation value of the number operator as a function of $\rho$ for the same values of the parameter $\nu$ as used in Figure 1. For small values of $\rho$, the expectation value of the number operators shows a linear increasing profile in agreement with the asymptotic behaviour (72). That is, the slop at $\rho = 0$ depends on the deformation parameter $\nu$. For $\nu > 0$ the expectation value is always below the undeformed case $\nu = 0$, which is a straight line given by $\langle N\rangle = \rho$. For $\nu < 0$ it is above that straight line. For large $\rho$ all graphs show the linear behavior as indicated in (72). That is, they exhibit a straight line parallel to the undeformed case. As the graphs do not intersect, we also conclude that the expectation value of the number operator for a fixed value of $\rho > 0$ is

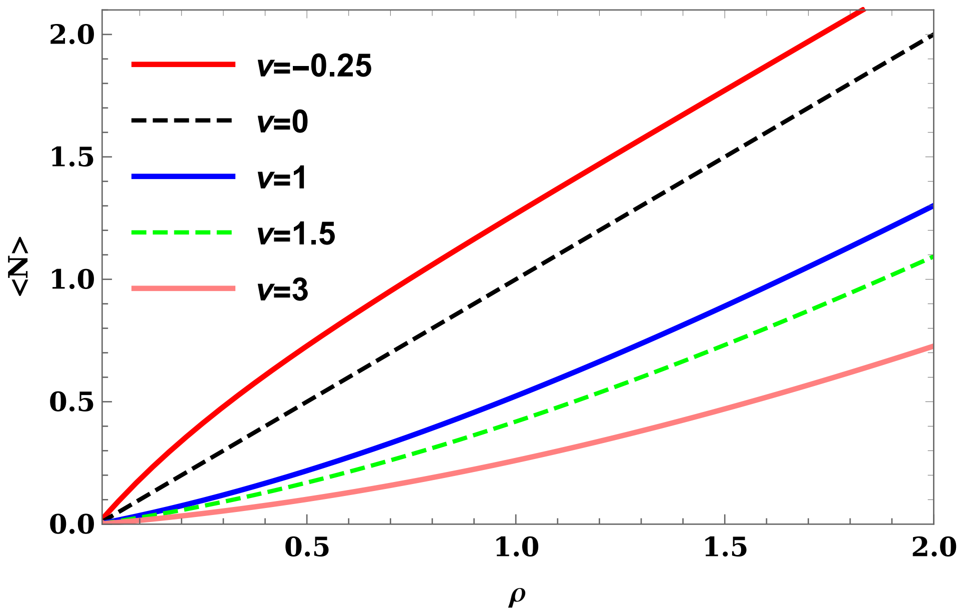


Figure 2: The expectation value of the number operator $\langle N\rangle$ as a function of $\rho = |\lambda|^2/\sqrt{1-\gamma^2}$ for same values of the deformation parameter $\nu$ as in figure 1.

a strictly decreasing function of the deformation parameter $\nu$.

Now we study the Mandel parameter, which is usually used to evaluate photon-counting statistics. The Mandel parameter $Q$ is defined as [49]

$$Q := \frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle} - 1 = \frac{\langle N(N-1)\rangle}{\langle N\rangle} - \langle N\rangle\,. \tag{73}$$

For Mandel parameters with values $Q = 0$, $Q < 0$ and $Q > 0$ the coherent states are Poissonian, sub-Poissonian (non-classical) and super-Poissonian, respectively [49, 50]. In other words, the Mandel parameter is a scale to qualify the deviations of a system from the classical Poissonian distribution. In order to evaluate $Q$ we first note that

$$\begin{aligned} N(N-1) &= \frac{H^2}{1-\gamma^2} - \frac{2(\nu+1)}{\sqrt{1-\gamma^2}}\,H + \left(\nu+\frac{1}{2}\right)\left(\nu+\frac{3}{2}\right), \\ H^2 &= (a^\dagger)^2(a)^2 + 2\sqrt{1-\gamma^2}\,a^\dagger a + \left(1-\gamma^2\right)\left(\nu^2+\nu R+\tfrac{1}{4}\right), \end{aligned} \tag{74}$$

resulting in the expectation value

$$\langle N(N-1)\rangle = (\rho-\nu)^2 + \left(\nu-\frac{1}{2}\right)^2 - \nu(2\nu+1)\langle R\rangle + 2\nu - \frac{1}{4}, \tag{75}$$

and Mandel parameter

$$Q = \frac{\nu(\nu+1) - \nu(1+2\rho)\langle R\rangle - \nu^2\langle R\rangle^2}{\rho-\nu+\nu\langle R\rangle}\,. \tag{76}$$

In figure 3 we show the behavior of the Mandel parameter for same values of the deformation parameter as in the previous figures. As expected, the undeformed case $\nu = 0$ exhibits the classical Poissonian behavior, i.e., we have $Q = 0$ for all values of $\rho$. However, for $\nu < 0$ we have $Q < 0$ for all $\rho > 0$ indicating sub-Poissonian photon statistics. On the other hand, for a positive deformation parameter $\nu > 0$ the behavior is super-Poissonian as then $Q > 0$ for all $\rho > 0$. For $\rho \to \infty$ we observe that the Mandel parameter asymptotically approaches the classical value $Q = 0$. That is, for large $\rho$ the deformation becomes less dominant and we see a more classical behavior approached from above and below for a positive and negative deformation, respectivly.

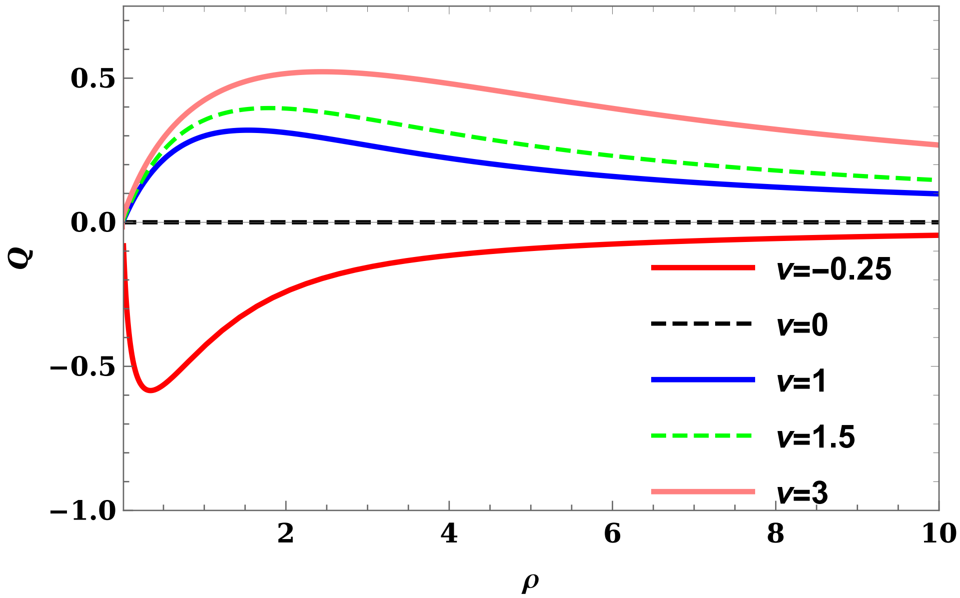


Figure 3: The Mandel parameter $Q$ as a function of $\rho$ for or same values of the deformation parameter as in before.

Finally, in order to study photon bunching and anti-bunching effects, we consider the normalised second-order correlation function defined via [55]

$$g^2 := \frac{\langle\lambda|N^2|\lambda\rangle}{\langle\lambda|N|\lambda\rangle^2} = \frac{(\rho-\nu)^2+\nu^2+\rho-2\nu^2\langle R\rangle}{(\rho-\nu+\nu\langle R\rangle)^2}\,. \tag{77}$$

This second-order correlation function can be used to quantify the quantum behaviours of the electromagnetic field. Whereas for perfect coherent light, where one has $g^2 = 1$, any so-called bunched light exhibit values $g^2 > 1$, and antibuched light has values $g^2 < 1$. In figure 4, we present graphs of $g^2$ for different values of the deformation parameter. We observe that $g^2 > 1$ for all values of $\rho$, indicating a bunching effect, which becomes less dominant when $\rho \to \infty$ as $g^2 \to 1$ in this limit. An anti-bunching effect, i.e. $g^2 < 1$, does not occur for any value of $\nu$.

## 6 Summary and Conclusion

In this work we studied the GWD harmonic oscillator in one dimension by using the generalised Dunkl derivative (2). We have found the explicit form of the energy eigenvalues and associated eigenfunctions. In fact, these spectral properties are closely related to the standard WD harmonic oscillator. This is most obvious when changing from the parameter set $\{\alpha,\beta,\gamma\}$ to the set $\{\mu,\nu,\gamma\}$. The energy eigenvalues then become independent of the "gauge" parameter $\mu$ and only depend on the remaining parameters $\nu$ and $\gamma$. The energy eigenfunctions are in essence expressed in terms of associated Legendre polynomials and a Gaussian function. The standard WD case is obtained when setting $\alpha = -\beta$ and $\gamma = 0$, that is $\mu = \nu = \alpha$, and we obtain in this limit the known spectral properties of the standard WD oscillator. We also presented for the first time an explicit expression for the quantum propagator. This result was even not presented before for the standard WD case.

We also constructed the coherent states of the GWD oscillator. Again these states have close similarities with those of standard WD quantum mechanics recently studied by Ghazouani [52], which can be obtain from the present ones in the limit $\alpha = -\beta$ and $\gamma = 0$. Nevertheless our results are more general, in particular, as

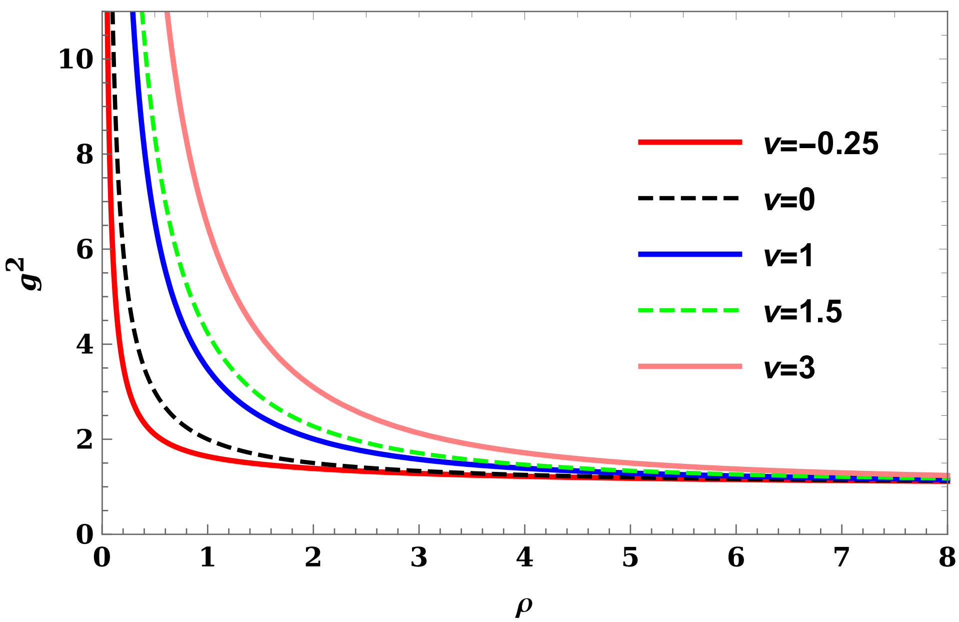


Figure 4: The second-order correlation function $g^2$ as a function of $\rho$ for same values of the deformation parameter as in before.

we allow for a negative deformation parameter $\nu$, which result in significant different behaviours as those for positive deformations. For example, the expectation value of the reflection operator $R$ was found to be always positive in cases where the deformation parameter $\nu$ is positive. Negative expectation values of $R$ can only be achieved when $-\frac{1}{2} < \nu < 0$ and $|\lambda|$ is sufficiently large. This is rather surprising as spec $R = \{-1, +1\}$.

In going forward, we presented here some non-classical properties of the GWD deformed coherent states such as photon statistics, Mandel parameter, and bunching or anti-bunching effect. It turns out that the probability $P_n^{(\nu)}$ of detecting $n$ photons in a coherent state does not follow a Poissonian distribution in general. For the case of a negative deformation $-\frac{1}{2} < \nu < 0$ this probability increases $P_n^{(\nu)} > P_n^{(0)}$, whereas for positive deformations $\nu > 0$ it is less than the undeformed probability $P_n^{(\nu)} < P_n^{(0)}$.

## Acknowledgments

The authors thank the referees for a thorough reading of the manuscript and constructive suggestions.

## Data Availability Statements

No data are associated in the manuscript.

## Appendix: The adjoint of the generalised Wigner-Dunkl derivative

In this appendix be provide a proof that $D_x^\dagger = -\left(\frac{1+\gamma R}{1-\gamma R}\right) D_x$ on $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}\mu)$. To begin with, let us consider two arbitrary states $\phi, \psi \in \mathcal{H}$ and the matrix element

$$
\begin{aligned}
\langle \phi | \partial_x \psi \rangle &= \int_{-\infty}^{+\infty} \mathrm{d}x\, |x|^{2\mu} \phi^*(x)\, (\partial_x \psi)(x) \\
&= \phi^*(x)\psi(x)\, |x|^{2\mu} \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x) \partial_x \left(\phi^*(x)\, |x|^{2\mu}\right) \\
&= -\int_{-\infty}^{+\infty} \mathrm{d}x\, |x|^{2\mu} \psi(x) \left( (\partial_x \phi^*)(x) + \frac{2\mu}{x} \phi^*(x) \right) .
\end{aligned}
\tag{78}
$$

Therefore, the adjoint of the partial derivative with respect to $x$ in the weighted Hilbert space $\mathcal{H}$ is given by

$$(\partial_x)^\dagger = -\partial_x - \frac{2\mu}{x} = -\partial_x - \frac{2(\alpha-\beta\gamma)}{(1-\gamma^2)x}\,. \tag{79}$$

Using this result together with $R^\dagger = R$ and $x^\dagger = x$ we find for the adjoint $D_x^\dagger$ following form:

$$\begin{aligned}
D_x^\dagger &= \left((1-\gamma R)\partial_x + \tfrac{\alpha}{x} + \tfrac{\beta}{x}R\right)^\dagger \\
&= \partial_x^\dagger(1-\gamma R)^\dagger + \left(\tfrac{\alpha}{x}\right)^\dagger + R^\dagger\left(\tfrac{\beta}{x}\right)^\dagger \\
&= \left(-\partial_x - \tfrac{2(\alpha-\beta\gamma)}{(1-\gamma^2)x}\right)(1-\gamma R) + \tfrac{\alpha}{x} + R\tfrac{\beta}{x} \\
&= -(1+\gamma R)\partial_x - \left(\tfrac{1+\gamma R}{1-\gamma R}\right)\tfrac{\alpha}{x} + \left(\tfrac{1+\gamma R}{1-\gamma R}\right)R\tfrac{\beta}{x} \\
&= -\left(\tfrac{1+\gamma R}{1-\gamma R}\right)\left[(1-\gamma R)\partial_x + \tfrac{\alpha}{x} + \tfrac{\beta}{x}R\right] \\
&= -\left(\tfrac{1+\gamma R}{1-\gamma R}\right)D_x.
\end{aligned}$$

Hence we obtain the relation (11).

## References


[1] M. Arik, D. Coon, J. Math. Phys. 17 (1976) 524.

[2] A. J. Macfarlane, J. Phys. A 22, 4581 (1989).

[3] L. Biedenharn, J. Phys. A 22 (1989) L 873.

[4] A. Lorek, A. Ruffing and J. Wess, Z. Phys. C 74, 369 – 377 (1997).

[5] W. S. Chung, K. S. Chung, J. Phys. A: Math. Gen. 24 (1991).

[6] W. S. Chung, Fortschr. Phys. 62, no. 8 (2014): 619-625.

[7] M. Chaichian, D. Ellinas, P. Kulish, Phys. Rev. Lett. 65, 980 (1990).

[8] M. Chaichian, D. Ellinas and P. Kulish, Chem. Phys. Lett. 302 (1999).

[9] W. S. Chung, J. Adv. Phys. 4, 1–4 (2015).

[10] J. Crnugelj, M. Martinis, V. Mikuta-Martinis, Phys. Lett. A 188, 347-354 (1994).

[11] H. Hassanabadi, P. Sedaghatnia, W. S. Chung, and S. Zarrinkamar. Int. J. Mod. Phys. A 34, no. 28 (2019): 1950165.

[12] L. Hellström, S. Silvestrov, Expo. Math. 23, no. 2 (2005): 99-125.

[13] P. Sedaghatnia, H. Hassanabadi, and M. de Montigny, Int. J. Mod. Phys. E, 29(08), 2050064 (2020).

[14] V. X. Genest, L. Vinet and A. Zhedanov, J. Phys. Conf. Ser. (2014) 512, 012010.

[15] V. X. Genest, M. E. H. Ismail, L. Vinet and A. Zhedanov, Commun. Math. Phys. 329 (2014): 999-1029.

[16] V. X. Genest, L. Vinet, and A. Zhedanov, SIGMA 9, 18 (2013).

[17] V. X. Genest, M. E. H. Ismail, L. Vinet and A. Zhedanov, J. Phys. A: Math. Theor. 46, 145201 (2013).

[18] E. Wigner, Phys. Rev. 77, 711 (1950).

[19] C. Dunkl, T. Am. Math. Soc. 311, 167 (1989).

[20] C. Dunkl, Math. Z. 197, 33 (1988).

[21] L. Vinet and J. F. Van Diejen, Calogero-Mosersutherland Models (Springer, 2000).

[22] G. J. Heckman, Prog. Math. 101, 181 (1991).

[23] P. Sedaghatnia, H. Hassanabadi, A. D. Alhaidari, and W. S. Chung. Int. J. Mod. Phys. A 37, no. 35 (2022): 2250223-117.

[24] F. Dai and Y. Xu, Approximation Theory and Harmonic Analysis on Spheres and Balls (Springer, 2013).

[25] W. S. Chung and H. Hassanabadi , Eur. Phys. J. Plus 136 (2021): 239.

[26] S. H. Dong, L. F. Quezada, W. S. Chung, P. Sedaghatnia, and H. Hassanabadi. Ann. Phys. (2023): 169259.

[27] S. H. Dong, W. H. Huang, W. S. Chung, P. Sedaghatnia, and H. Hassanabadi. Eur. phys. Lett. 135, no. 3 (2021): 30006.

[28] S. H. Dong, W. H. Huang, P. Sedaghatnia, H. Hassanabadi, Results Phys. 34 (2022): 105294.

[29] S. Hassanabadi, J. Kříž, B. C. Lütfüoğlu, and H. Hassanabadi. Phys. Scr. 97, no. 12 (2022): 125305.

[30] W. S. Chung, H. Hassanabadi, Eur. Phys. J. Plus 136 (2) (2021) 1-11.

[31] S. Ghazouani, I. Sboui, M.A. Amdouni, M.B. El Hadj Rhouma, J. Phys. A 52 (2019): 225202.

[32] S. Ghazouani, I. Sboui, J. Phys. A 53 (2019): 035202.

[33] M. Salazar-Ramírez, D. Ojeda-Guillén, R.D. Mota, V.D. Granados, Eur. Phys. J. Plus 132 (2017): 39.

[34] M. Salazar-Ramírez, D. Ojeda-Guillén, R.D. Mota, V.D. Granados, Mod. Phys. Lett. A 33 (2018): 1850112.

[35] R. D. Mota, D. Ojeda-Guillén, M. Salazar-Ramírez, V. D. Granados, Mod. Phys. Lett. A 36, no. 23 (2021): 2150171.

[36] S. Hassanabadi, P. Sedaghatnia, W. S. Chung, B. C. Lütfüoğlu, J. Kříž, H. Hassanabadi, Eur. Phys. J. Plus 138, no. 4 (2023): 1-7.

[37] M. R. Ubriaco, Phys. A: Stat. Mech, 414 (2014): 128-136.

[38] F. Merabtine, B. Hamil, B.C. Lütfüoğlu, A. Hocine, and M. Benarous, J. Stat. Mech.: Theory Exp., no. 5 (2023): 053102.

[39] B. Hamil, and B.C. Lütfüoğlu, Phys. A: Stat. Mech, 623 (2023): 128841.

[40] B. Hamil, and B.C. Lütfüoğlu, Eur. Phys. J. Plus 137, no. 7 (2022): 1-8.

[41] B. Hamil, and B.C. Lütfüoğlu, Few-Body Syst. 63, no. 4 (2022): 74

[42] B. Hamil, and B.C. Lütfüoğlu, Eur. Phys. J. Plus 137 (2022): 1241.

[43] V. V. Eremin, and A. A. Meldianov. Theor. Math. Phys, 147 (2006): 709-715.

[44] K. Nakamura, and K. Nakamura, Quantum Phononics: Introduction to Ultrafast Dynamics of Optical Phonons (2019): 25-50.

[45] W. S. Chung, G. Junker, S. H. Dong and H. Hassanabadi, Eur. Phys. Lett. 141 (2023) 32001.

[46] W. S. Chung, H. Hassanabadi, Mod. Phys. Lett. A 34 (2019): 1950190.

[47] H. Gohberg, S. Goldberg, and M. A. Kaashoek, Operator theory: Advances and applications. Classes of linear operators, 49 (1992).

[48] W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics. 3rd Edition, Springer, Berlin (1966).

[49] W. S. Chung, and H. Hassanabadi. Few-Body Syst. 62, no. 2 (2021): 24.

[50] W. S. Chung, B. C. Lütfüoğlu, and H. Hassanabadi. Int. J. Theor. Phys. 60 (2021): 1109-1126.

[51] M. Rösler, Comm. Math. Phys. 192 (1998): 519-542.

[52] S. Ghazouani, J. Phys. A 55 (2022):505203.

[53] J. R. Klauder and B. S. Skagerstam, World Scientific, (1985).

[54] J. Segura, J. Math. Anal. Appl. 374 (2011): 516–528.

[55] R. Loudon, Phys. Bull. 27 (1976): 21-23.